Journal of Sound and Vibration (2000) **233**(4), 727–738
doi:10.1006/jsvi.1999.2825, available online at http://www.idealibrary.com on IDEAL®

LETTERS TO THE EDITOR

AXISYMMETRIC VIBRATION OF INHOMOGENEOUS CLAMPED CIRCULAR PLATES: AN UNUSUAL CLOSED-FORM SOLUTION

I. ELISHAKOFF

Department of Mechanical Engineering, Florida Atlantic University, Boca Raton, Fl 33431-0991, *U.S.A.*

(*Received* 10 *December* 1999)

1. INTRODUCTION

The free vibrations of circular plates attracted investigators nearly two centuries ago [1, 2]. The vibrations of uniform plates have been studied by now quite extensively [3–6]. Although the circular plates of variable thickness received much less attention than the uniform plates, still, they have been investigated quite intensively. An *exact* solution was derived by Conway *et al.* [7] by noting an analogy [8, 9] that exists between the free vibration of truncated-cone beams and linearly tapered plates for the special case when the Poisson ratio equals one third. Convey *et al.* [7] derived natural frequencies for clamped tapered circular plates. Series of solutions have been provided by Soni [10] for plates with quadratically varying thickness. The method of Frobenius was utilized by Jain [11]. Frequency parameters of clamped and simply supported plates were computed, for the first two modes, for various values of a taper parameter and in-plane force, both for linear and parabolic variations of thickness. A perturbation method based on a small parameter was employed by Yang [12]. Zeroth- and first order asymptotic solutions were obtained for the natural frequencies of a clamped plate with linearly varying thickness. Lenox and Conway [13] obtained an exact expression for the buckling mode for the annular plate; at the later stage they performed the numerical computerized calculations for natural frequencies.

Most of the reported solutions utilized approximate techniques. Laura *et al.* [14] used the Rayleigh–Ritz method with polynomial co-ordinate functions that identically satisfy the external boundary conditions to study the vibrations and elastic stability of polar orthotropic circular plates of linearly varying thickness. Lal and Gupta [15] utilized Chebychev polynomials to obtain the frequencies of polar orthotropic annular plates of variable thickness. Gorman [16] and Kanaka Raju [17] employed the most universal technique, that of the finite element method for studying the dynamic behavior of polar orthotropic annular plates of variable thickness.

As far as the exact solutions are concerned, one very important paper should be mentioned. Harris [18] obtained a *closed-form* solution for both the mode shapes and the natural frequencies of the circular plate that is free at its edge. The stiffness was given as $D(r) = D_0(1 - r^2/R^2)^3$, where D_0 is the stiffness at the center, r the polar co-ordinate and R the radius.

The present note shares, with the paper by Harris [18], the property of offering *exact*, *closed-form* solution the natural frequency. However, whereas Harris [18] solves a *direct* vibration problem, we pose the *inverse* problem. Namely, we postulate the vibration mode and pursue the following objective: find the variation of the stiffness that leads to the postulated vibration mode. In some circumstances such a problem has a unique solution with a remarkable by-product: a closed-form expression for the natural frequency. This

0022-460X/00/240727 + 12 $35.00/0

paper complements that of Harris [18], who derived a solution for the circular plates that are free at the boundary, while the present note deals with clamped plates.

The adjective "unusual" utilized in the title was borrowed from the note by Convey [19], who derived the *closed-form* expression for the unsymmetrical bending of a particular circular plate resting on a Winkler foundation. He mentioned: "What is remarkable about this solution is that it is in a closed-form which is far simpler than the constant thickness disk solution which involves Kelvin functions." We consider a vibration case and likewise derive *closed-form* solution. Yet, it appears that the present solution is superior to that by Lenox and Convey [13] for here we derive not only the closed-form expression for the *displacement*, but the *natural frequency* too. The unusual aspect characteristic of reference [19] is preserved in our study: while for the exact solution for homogeneous circular plates one has the Bessel functions involved, in the inhomogeneous case the elementary functions suffice.

2. BASIC EQUATIONS

The differential equation that governs the free non-axisymmetric vibrations of the circular plate with variable thickness reads [20]

$$D(r)r^3\Delta\Delta W + \frac{\mathrm{d}D}{\mathrm{d}r}\left(2r^3\frac{\mathrm{d}^3W}{\mathrm{d}r^3} + r^2(2+\nu)\frac{\mathrm{d}^2W}{\mathrm{d}r^2} - r\frac{\mathrm{d}W}{\mathrm{d}r}\right)$$

$$+\frac{\mathrm{d}^2D}{\mathrm{d}r^2}\left(r^3\frac{\mathrm{d}^2W}{\mathrm{d}r^2} + \nu r^2\frac{\mathrm{d}W}{\mathrm{d}r}\right) - \rho h\omega^2 r^3 W = 0, \quad (1)$$

while Δ is the Laplace operator in polar co-ordinates,

$$\Delta = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}, \quad (2)$$

D is the bending stiffness, assumed to vary along the radial co-ordinate r,

$$D = D(r) = \frac{Eh^3}{12(1-\nu^2)}, \quad (3)$$

h is the thickness, ν the Poisson ratio, ρ the material density, r the radial co-ordinate, φ the circumferential co-ordinate, and W the mode shape. The Poisson ratio ν is assumed to be constant. Note that the governing equation reported by Kovalenko [20] is multiplied here by the term r^3, for further convenience. We are looking for the case when the inertial term

$$\delta(r) = \rho h \quad (4)$$

varies along r; likewise the stiffness is a function of r. We are interested in finding the *closed-form* solution for the natural frequency ω.

We pose the problem as an *inverse* vibration study. Note first that for the uniform circular plate that is under uniform load q_0 the displacement can be put in the form [21]

$$w = \frac{q_0}{64D}(R^2 - r^2)^2, \quad (5)$$

where R is the outer radius of the plate. We are interested in determining such a variation of $D(r)$ in equation (1) that the function

$$W(r) = (R^2 - r^2)^2 \tag{6}$$

serves as an exact mode shape. We confine our interest to circular plates that are clamped along the boundary $r = R$.

3. METHOD OF SOLUTION

We assume that the inertial term is represented as a polynomial,

$$\delta(r) = \sum_{i=0}^{m} a_i r^i. \tag{7}$$

Since $W(r)$ is the fourth order polynomial expression in terms of r, in view of equation (7) the last term in equation (1) is the polynomial expression of order $m + 7$. Since the operator $\Delta\Delta$ in equation (1) involves the four-fold differentiation with respect to r, in order for the highest degree of the first term's polynomial expression in $Dr^3\Delta\Delta W$ to be of order $m + 7$, it is necessary and sufficient for the stiffness to be represented as a polynomial of degree $m + 4$. Thus, the sought stiffness can be put in the form

$$D(r) = \sum_{i=0}^{m+4} b_i r^i. \tag{8}$$

Further steps involve the substitution of equations (6)–(8) into the governing differential equation (1) and demanding the so-obtained polynomial expression to vanish. This implies that all the coefficients in front of powers r^i must be zero, leading, in turn, to the set of algebraic equations in terms of b_i and ω^2. We consider various variations for the inertial term $\delta(r)$ in equation (4).

4. CONSTANT INERTIAL TERM ($m = 0$)

In this case, the stiffness is sought as a fourth order polynomial,

$$D(r) = b_0 + b_1 r + b_2 r^2 + b_3 r^3 + b_4 r^4. \tag{9}$$

We get instead of the differential equation (1), the equation

$$\sum_{i=0}^{7} c_i r^i = 0, \tag{10}$$

where

$$c_0 = 0, \qquad c_1 = 0, \qquad c_2 = -4(1 + \nu)R^2 b_1, \qquad c_3 = 64b_0 - 16(1 + \nu)R^2 b_2 - a_0\omega^2 R^4,$$

$$c_4 = 12(11 + \nu)b_1 - 36(1 + \nu)R^2 b_3, \qquad c_5 = 32(7 + \nu)b_2 - 64(1 + \nu)R^2 b_4 + 2\alpha_0\omega^2 R^2,$$

$$c_6 = (340 + 60\nu)b_3, \qquad c_7 = 96(5 + \nu)b_4 - a_0\omega^2. \tag{11}$$

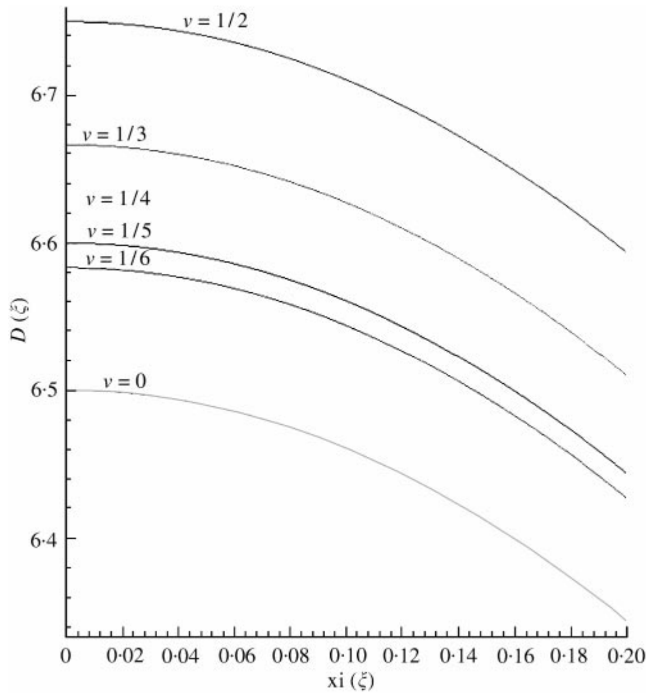

Figure 1. Variation of the stiffness of the clamped circular plate with constant inertial term, when the Poisson ratio takes the values 0, $\frac{1}{6}$, $\frac{1}{5}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$.

Since the left-hand side of the differential equation (10) must vanish for any r within $[0; R]$, we demand that all the coefficients c_i to be zero. This leads to a homogeneous set of six linear algebraic equations for six unknowns. It turns out that the determinant of the matrix of the set derived from equation (11) is identically zero. Therefore, a non-trivial solution is obtainable. From the requirement $c_7 = 0$, the natural frequency squared is obtained as

$$\omega^2 = 96(5 + \nu)b_4/a_0. \tag{12}$$

Upon substitution of equation (12) into equation (11), the remaining equations yield the coefficients in the stiffness:

$$b_0 = \frac{13 + \nu}{2} R^4 b_4, \qquad b_1 = 0, \qquad b_2 = -4R^2 b_4, \qquad b_3 = 0. \tag{13}$$

Hence, the stiffness reads

$$D(r) = \left(\frac{13 + \nu}{2} R^4 - 4R^2 r^2 + r^4\right) b_4. \tag{14}$$

Figure 1 depicts the stiffness for various values of the Poisson ratio ν.

5. LINEARLY VARYING INERTIAL TERM ($m = 1$)

Instead of the set (11) we get here seven linear algebraic equations with seven unknowns:

$$-4(1+\nu)R^2 b_1 = 0, \qquad 64b_0 - 16(1+\nu)R^2 b_2 - a_0\omega^2 R^4 = 0,$$

$$12(11+\nu)b_1 - 36(1+\nu)R^2 b_3 - a_1\omega^2 R^4 = 0,$$

$$32(7+\nu)b_2 - 64(1+\nu)R^2 b_4 + 2a_0\omega^2 R^2 = 0,$$

$$(340+60\nu)b_3 - 100(1+\nu)R^2 b_5 + 2a_1\omega^2 R^2 = 0,$$

$$96(5+\nu)b_6 - a_0\omega^2 = 0, \qquad (644+140\nu)b_5 - a_1\omega^2 = 0. \tag{15}$$

In order to have a non-trivial solution the determinant of the set (15),

$$(1+\nu)(7+\nu)(13765 + 5643\nu + 728\nu^2 + 30\nu^3)a_1 = 0 \tag{16}$$

must vanish, leading to $a_1 = 0$. Yet this result signifies that the linear inertial coefficient is a constant. Thus, the problem solved in the previous section is reobtained.

6. PARABOLICALLY VARYING INERTIAL TERM ($m = 2$)

For $m = 2$, i.e., the plate whose material density varies parabolically,

$$\delta(r) = a_0 + a_1 r + a_2 r^2, \tag{17}$$

the bending stiffness has to be sought as the sixth order polynomial:

$$D(r) = b_0 + b_1 r + b_2 r^2 + b_3 r^3 + b_4 r^4 + b_5 r^5 + b_6 r^6. \tag{18}$$

Substitution of equation (6) in conjunction with equations (17) and (18) into the governing differential equation (1) yields

$$\sum_{i=0}^{9} d_i r^i = 0, \tag{19}$$

where

$$d_0 = 0, \qquad d_1 = 0, \qquad d_2 = -4R^2(1+\nu)b_1, \qquad d_3 = 64b_0 - 16R^2(1+\nu)b_2 - a_0\omega^2 R^4,$$

$$d_4 = 12(11+\nu)b_1 - 36R^2(1+\nu)b_3 - a_1\omega^2 R^4,$$

$$d_5 = 32(7+\nu)b_2 - 64R^2(1+\nu)b_4 + \omega^2(2a_0R^2 - a_2R^4),$$

$$d_6 = 20(17+3\nu)b_3 - 100R^2(1+\nu)b_5 + 2a_1\omega^2 R^2,$$

$$d_7 = 96(5+\nu)b_4 - 144R^2(1+\nu)b_6 - \omega^2(a_0 - 2a_2R^2),$$

$$d_8 = (644+140\nu)b_5 - a_1\omega^2, \qquad d_9 = 64(13+3\nu)b_6 - a^2\omega^2. \tag{20}$$

As in the case of the constant inertial term, we demand that all $d_i = 0$. Thus, we get a set of eight equations with eight unknowns (seven coefficients b_i and ω^2). The resulting determinantal equation is

$$(1 + v)(7 + v)a_1(178945 + 114654v + 26393v^2 + 2574v^3 + 90v^4) = 0. \tag{21}$$

In order for the homogeneous system to possess a non-trivial solution we must demand the coefficient a_1 to vanish. We substitute $a_1 = 0$ into the set (20). For the natural frequency we arrive at the expression, obtainable from the requirement $d_9 = 0$,

$$\omega^2 = 64(13 + 3v)b_6/a_2. \tag{22}$$

Then, the coefficients in the stiffness are obtained as

$$b_0 = \frac{R^4 b_6}{12} \frac{(295 + 353v + 61v^2 + 3v^3)R^2 a_2 + (4732 + 2132v + 292v^2 + 12v^3)a_0}{(35 + 12v + v^2)a_2},$$

$$b_1 = 0, \qquad b_2 = \frac{R^2 b_6}{3} \frac{(295 + 58v + 3v^2)R^2 a_2 - (728 + 272v + 24v^2)a_0}{(35 + 12v + v^2)a_2},$$

$$b_3 = 0, \qquad b_4 = -\frac{b_6}{6} \frac{(95 + 15v)R^2 a_2 - (52 + 12v)a_0}{(5 + v)a_2}, \quad b_5 = 0, \tag{23}$$

were b_6 is an arbitrary constant. In order the natural frequency squared to be a positive quantity we demand that the ratio b_6/a_2 be positive. We have two sub-cases: (1) both b_6 and a_2 are positive and (2) both are negative. In the former case (1) the necessary condition $b_0 \geqslant 0$ for positivity of the stiffness $D(r)$ is identically satisfied. In the latter case the above inequality reduces to

$$(295 + 353v + 61v^2 + 3v^3)R^2 a_2 + (4732 + 2132v + 292v^2 + 12v^3)a_0 \geqslant 0, \tag{24}$$

leading to the inequality

$$\frac{a_0}{|a_2|R^2} \leqslant \frac{295 + 353v + 61v^2 + 3v^3}{4732 + 2132v + 292v^2 + 12v^3}. \tag{25}$$

One can immediately see that

$$\frac{a_0}{|a_2|R^2} < 1. \tag{26}$$

Hence, the associated variation of the inertial coefficient

$$\delta(r) = a_0 + a_2 r^2 \tag{27}$$

takes a negative value at $r = R$. Thus, the possibility that both a_2 and b_6 be negative should be discarded as physically unrealizable. We conclude, therefore, that in equation (22) both a_2 and b_6 must constitute positive quantities.

7. CUBIC INERTIAL TERM ($m = 3$)

For $m = 3$, the following set of nine linear algebraic equations with nine unknowns is obtained:

$$-4R^2(1+v)b_1 = 0, \qquad 64b_0 - 16R^2(1+v)b_2 - a_0\omega^2R^4 = 0,$$

$$12(11+v)b_1 - 36R^2(1+v)b_3 - a_1\omega^2R^4 = 0,$$

$$32(7+v)b_2 - 64R^2(1+v)b_4 + \omega^2(2a_0R^2 - a_2R^4) = 0,$$

$$20(17+3v)b_3 - 100R^2(1+v)b_5 + \omega^2(2a_1R^2 - a_3R^4) = 0,$$

$$96(5+v)b_4 - 144R^2(1+v)b_6 - \omega^2(a_0 - 2a_2R^2) = 0,$$

$$(644+140v)b_5 - 196(1+v)R^2b_7 - \omega^2(a_1 - 2a_3R^2) = 0,$$

$$(832+192v)b_6 - a_2\omega^2 = 0, \qquad (1044+252v)b_7 - a_3\omega^2 = 0. \tag{28}$$

The determinantal equation stemming from its reads

$$(1+v)(7+v)M = 0, \tag{29}$$

where

$$M = (490685R^2 + 791887v + 367095v^2 + 71999v^3 + 6316v^4 + 210v^5)R^2a_3$$

$$+ (5189405 + 4577581v + 1567975v^2 + 259397v^3 + 20628v^4 + 630v^5)a_1. \tag{30}$$

The solution of equation (29) is

$$a_1 = -\frac{(7549 + 8931v + 1452v^2 + 70v^3)a_3}{79837 + 36033v + 4916v^2 + 210v^3}. \tag{31}$$

Upon substitution of equation (28) into all equations of set (25), the natural frequency squared equals

$$\omega^2 = \frac{36(29+7v)b}{a_3} \tag{32}$$

and we get the following solution for the coefficients in the stiffness:

$$b_0 = 3[(8555 + 12302v + 4240v^2 + 514v^3 + 21v^4)R^2a_2 + (137228 + 94952v$$

$$+ 23392v^2 + 2392v^3 + 84v^4)a_0]R^4b_7/64a_3(455 + 261v + 49v^2 + 3v^3),$$

$$b_1 = 0,$$

$$b_2 = 3[(8555 + 3747v + 493v^2 + 21v^3)R^2a_2 - (21112 + 12984v + 2600v^2 + 168v^3)a_0]R^2b_7/16a_3(455 + 261v + 49v^2 + 3v^3),$$

$$b_3 = \frac{(7549 + 1382v + 70v^2)R^4b_7}{2753 + 578v + 30v^2}, \tag{33}$$

$$b_4 = \frac{2b_7[(1508 - 712v + 84v^2)a_0 - (2755 + 1100v + 105v^2)R^2a_2]}{32a_3(65 + 28v + 3v^2)},$$

$$b_5 = -\frac{2R^2(4255 + 832v + 42v^2)b_7}{2753 + 578v + 30v^2}, \qquad b_6 = \frac{9(29 + 7v)a_2b_7}{16a_3(13 + 3v)},$$

where b_7 is an arbitrary constant. For the particular case $v = \frac{1}{3}$, the stiffness equals

$$D(r) = \left[\frac{2773}{2464}\frac{R^6a_7}{a_3} + \frac{235}{16}\frac{R^4a_0}{a_3} + \left(\frac{8319}{2464}\frac{R^4a_2}{a_3} - \frac{141}{112}\frac{R^2a_0}{a_3}\right)r^2 + \frac{72157}{26541}R^4r^3 + \left(-\frac{3525}{896}\frac{R^2a_2}{a_2} + \frac{141}{64}\frac{a_0}{a_3}\right)r^4 - \frac{9074}{2949}R^2r^5 + \frac{141}{112}\frac{a_2}{a_3}r^6 + r^7\right]b_7. \tag{34}$$

8. GENERAL INERTIAL TERM ($m \geqslant 4$)

Consider now the general expression of the inertial term given in equation (7), and the stiffness in equation (8), for $m \geqslant 4$. Substitution of equation (6)–(8) into the terms of the differential equation yields

$$r^3D(r)\Delta\Delta W = -64r^3\sum_{i=0}^{m+4} b_ir^i, \tag{35}$$

$$\frac{dD}{dr}\left(2r^3\frac{d^3W}{dr^3} + r^2(2 + v)\frac{d^2W}{dr^2} - r\frac{dW}{dr}\right) = 4[(17 + 3v)r^2 - R^2(1 + v)]r^2\sum_{i=1}^{m+4} ib_ir^{i-1}, \tag{36}$$

$$\frac{d^2D}{dr^2}\left(r^3\frac{d^2W}{dr^2} + vr^2\frac{dW}{dr}\right) = 4[(3 + v)r^2 - R^2(1 + v)]r^3\sum_{i=2}^{m+4} i(i-1)b_ir^{i-2}, \tag{37}$$

and

$$\rho h\omega^2r^3W = \omega^2r^3(R^2 - r^2)^2\sum_{i=0}^{m} a_ir^i. \tag{38}$$

Demanding the sum of equations (41)–(44) to be zero, we obtain the equation

$$\sum_{i=0}^{m+7} g_ir^i = 0, \tag{39}$$

where the coefficients g, are

$$g_0 = 0, \qquad g_1 = 0, \qquad g_2 = -4(1+v)b_1, \tag{40, 41}$$

$$g_3 = 64b_0 - 16R^2(1+v)b_2 - a_0R^4\omega^2, \tag{42}$$

$$g_4 = 12(11+v)b_1 - 36R^2(1+v)b_3 - a_1R^4\omega^2 \tag{43}$$

$$g_5 = 32(7+v)b_2 - 64R^2(1+v)b_4 - \omega^2(a_2R^4 - 2a_0R^2), \tag{44}$$

$$g_6 = 20(17+3v)b_3 - 100R^2(1+v)b_5 - \omega^2(a_3R^4 - 2a_1R^2), \tag{45}$$

$$\vdots$$

for $7 \leqslant i \leqslant m+3$,

$$g_i = [64 + 4(i-1)(17+3v) + 4(i-2)(i-1)(3+v)]b_{i-1} - 4(i+1)^2R^2(1+v)b_{i+1}$$
$$- \omega^2(a_{i-1}R^4 - 2a_{i-3}R^2 + a_{i-5}), \tag{46}$$

$$\vdots$$

$$g_{m+4} = [64 + 4(m+1)(17+3v) + 4m(m+1)(3+v)]b_{m+1} - 4(m+3)^2R^2b_{m+3}$$
$$- \omega^2(-2a_{m-1}R^2 + a_{m-3}), \tag{47}$$

$$g_{m+5} = [64 + 4(m+2)(17+3v) + 4(m+1)(m+2)(3+v)]b_{m+2} - 4(m+4)^2R^2b_{m+4}$$
$$- \omega^2(-2a_mR^2 + a_{m-2}), \tag{48}$$

$$g_{m+6} = [64 + 4(m+3)(17+3v) + 4(m+2)(m+3)(3+v)]b_{m+3} - \omega^2a_{m-1}, \tag{49}$$

$$g_{m+7} = [64 + 4(m+4)(17+3v) + 4(m+2)(m+3)(3+v)]b_{m+4} - \omega^2a_m, \tag{50}$$

We demand all coefficients g_i to be zero; thus, we get a set of $m+6$ homogeneous linear algebraic equations for $m+6$ unknowns. In order to find a non-trivial solution the determinant of set (40)–(50) must vanish. We expand the determinant along the last column of the matrix of the set, getting a linear algebraic expression with the coefficients a_i as coefficients. The determinantal equation yields a condition for which the non-trivial solution is obtainable. In this case the general expression of the natural frequency squared is obtained from the equation $g_{m+7} = 0$, resulting in

$$\omega^2 = [64 + 4(m+4)(17+3v) + 4(m+3)(m+4)(3+v)]b_{m+4}/a_m. \tag{51}$$

Note that the formulae pertaining the cases $m = 0$, 2 and 3 are *formally* obtainable from equation (51) by appropriate substitution.

9. ALTERNATIVE MODE SHAPES

Let us pose now the following question. In previous sections we postulated the expression given in Equation (6), which is proportional to the deflection of uniform circular plates under distributed loading. Equation (6) represents a fourth order polynomial. A natural question arises: can an inhomogeneous circular plate possess a simpler expression? A simplest polynomial expression, that satisfies the boundary conditions is

$$\psi(r) = (R - r)^2 \tag{52}$$

which represents a second order polynomial. The third order polynomial

$$\psi(r) = (R - r)^3, \tag{53}$$

as well as the fourth order polynomial,

$$\psi(r) = (R - r)^4, \tag{54}$$

also satisfy the boundary conditions. Note that equation (54) is also a fourth order polynomial, as in equation (6) although they are different. Expressions (52)–(54) are the candidate functions for both the Rayleigh–Ritz or Bubnov–Galerkin methods. Thus, in essence, we ask if the co-ordinate functions utilizable for *approximate* evaluation of natural frequencies of either homogeneous or inhomogeneous plates, can serve as *exact* buckling modes. We consider the candidate mode shape given in equation (52).

9.1. PARABOLIC MODE SHAPE

Substitution of equation (52) into differential equation (1) in conjunction with equation (9) for constant inertia term ($m = 0$) yields the equation

$$\sum_{i=0}^{7} e_i r^i = 0, \tag{55}$$

where

$$e_0 = -2b_0R, \qquad e_1 = 0, \qquad e_2 = 2(1+\nu)b_1 + 2R(1-2\nu)b_2,$$

$$e_3 = 8(1+\nu)b_2 + 4R(1-3\nu)b_3 - a_0R^2\omega^2,$$

$$e_4 = 18(1+\nu)b_3 - 6R(1-4\nu)b_4 + 2a_0R^2\omega^2,$$

$$e_5 = 32(1+\nu)b_4 - a_0\omega^2, \qquad e_6 = 0, \qquad e_7 = 0. \tag{56}$$

In order for equation (55) to be valid for every r, we require $e_i = 0$, for i taking values 0, 2, 3, 4, 5, for the remaining requirements are identically satisfied. We get five equations for six unknowns. Taking b_4 to be an arbitrary constant we get the expression for the natural frequency squared,

$$\omega^2 = 32(1+\nu)b_4/a_0, \tag{57}$$

with attendant stiffness coefficients

$$b_0 = 0, \qquad b_1 = \frac{R^3(-107 + 155v + 106v^2 + 24v^3)b_4}{18(1 + 3v + 3v^2 + v^3)},$$

$$b_2 = \frac{R^2(107 + 59v + 12v^2)b_4}{18(1 + 2v + v^2)}, \qquad b_3 = -\frac{5R(7 + 4v)b_4}{9(1 + v)}. \tag{58}$$

The necessary condition for non-negativity of the stiffness is $b_1 \geqslant 0$. The roots of the equation $b_1 = 0$ are

$$v_1 = \tfrac{1}{2}, \qquad v_2 = (-59 + \mathrm{i}\sqrt{1655})/24, \qquad v_3 = (-59 - \mathrm{i}\sqrt{1655})/24. \tag{59}$$

The last two roots, as complex numbers, have no physical significance. Only the first root, corresponding to incompressible material, is acceptable. The associated expression for the stiffness is

$$D(r) = \left(\frac{31R^2}{9} r^2 - \frac{10R}{3} r^3\right) b_4. \tag{60}$$

Figure 2 depicts the variation of the stiffness. The candidate mode shapes given in equation (53) and (54) should be investigated separately. It is conjectured that an polynomial function, that satisfies the boundary conditions, may arbitrary correspond to a physically realizable material density and/or stiffness distribution. For example, for the Poisson's ratio that differs from $\frac{1}{2}$, the plate does not possess the parabolic shape given in equation (52).

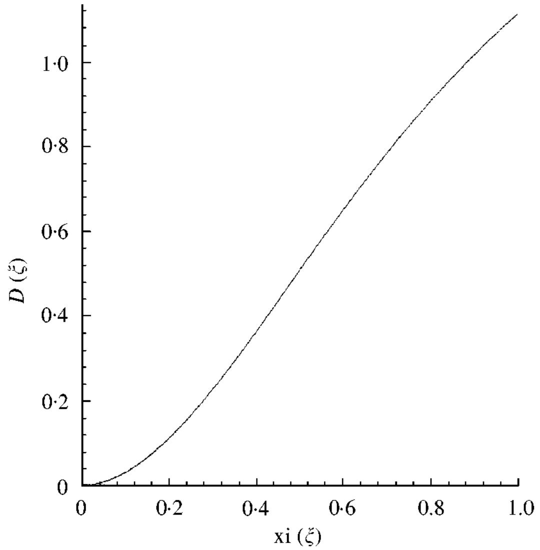

Figure 2. Variation of the stiffness of the clamped incompressible ($v = 0{\cdot}5$) circular plate with alternative parabolic mode shape.

10. CONCLUSION

Exact, closed-form solutions have been obtained here, apparently for the first time in the literature, for the inhomogeneous circular plates clamped along their boundary.

ACKNOWLEDGMENT

Help provided by Mr. Z. Guédé of IFMA (Aubière, France) is gratefully appreciated. I. Elishakoff acknowledges support by the National Science Foundation (Program Director: Dr. K. P. Chong) through the grant 99-10195. Opinions, findings and conclusions expressed are those by the writer and do not necessarily reflect the views of the sponsor.

REFERENCES

1. E. F. F. CHLADNI, 1803 *Die Akustik*, Leipzig.
2. S. D. POISSON 1829 *Memories de l'Academie Royales des Sciences de l'Institut de la France*, Ser. 2, Vol: **8**, 357. L'Équilibre et le mouvement des corps élastiques.
3. V. S. GONTKEVICH 1964 *Free Vibrations of Plates and Shells*. Kiev. Naukova Dumka Publishing. (in Russian).
4. A. W. LEISSA 1969 *NASA SP* 160. Vibration of plates.
5. A. W. LEISSA 1981 *The Shock and Vibration Digest* **10**, 19–36. Plate vibration research: 1976–1980.
6. A. W. LEISSA 1987 *The Shock and Vibration Digest* **19**, 10–24. Recent studies in plate vibrations: 1981–85, Part II, complicating effects.
7. H. D. CONWAY, E. C. H. BECKER and J. F. DUBIL 1964 *Journal of Applied Mechanics* **31**, 329–331. Vibration frequencies of tapered bars and circular plates.
8. H. D. CONVEY 1957 *Zeitschrift fuer angewandte Mathematik und Mechanik* **37**, 406–407. An analogy between the flexural vibrations of a cone and a disc of linearly varying thickness.
9. H. D. CONVEY 1958 *Ingenieur-Archiv* **26**, 408–410. Some special solutions for flexural vibrations of disc of varying thickness.
10. S. R. SONI 1972 *Ph.D. thesis*, *University of Roorkee*. Vibrations of elastic plates and shells of variable thickness.
11. R. K. JAIN 1972 *Journal of Sound and Vibration* **23**, 406–414. Vibrations of circular plates of variable thickness under an inplane force.
12. J. S. YANG 1993 *Journal of Sound and Vibration* **165**, 178–184. The vibration of a circular plate with varying thickness.
13. T. A. LENOX and H. D. CONVEY 1980 *Journal of Sound and Vibration* **68**, 231–239. An exact, closed-form solution for the flexural vibration of a thin annular plate having a parabolic thickness variation.
14. P. A. A. LAURA, D. R. AVALOS and C. D. GALLES 1982 *Journal of Sound and Vibration* **82**, 151–156. Vibrations and elastic stability of polar orthotropic circular plates of linearly varying thickness.
15. R. LAL and U. S. GUPTA 1982 *Journal of Sound and Vibration* **83**, 229–240. Axisymmetric vibrations of polar orthotropic annular plates of variable thickness.
16. D. G. GORMAN 1983 *Journal of Sound and Vibration* **86**, 47–60. Natural frequencies of transverse vibration of polar orthotropic variable thickness annular plate.
17. K. KANAKA RAJU 1977 *Journal of Sound and Vibration* **50**, 399–403. Large amplitude vibrations of circular plates with varying thickness.
18. C. Z. HARRIS 1968 *Quarterly Journal of Mechanics and Applied Mathematics* **21**, 320–327. The normal modes of a circular plate of variable thickness.
19. H. D. CONVEY 1980 *Journal of Applied Mechanics* **47**, 204. An unusual closed-form solution for a variable-thickness plate on an elastic foundation.
20. A. D. KOVALENKO 1959 *Circular Plates of Variable Thickness*. Kiev: Naukova Dumka Publishing (in Russia).
21. S. TIMOSHENKO and S. WOINOWSKY-KRIEGER 1959 *Theory of Plates and Shells*. New York: McGraw-Hill Book Company.